We can approximately replace the infinite system (2.10) by a finite system of 11 equations with 11 unknowns. This finite system was solved several times in application to distinct values of the parameter ε. The results of the calculations are presented in Table 1.

Knowing the coefficients A_{2n}, we can easily find the quantities c and $p(x)$. Presented below are values of the coefficient γ (formula (3.2)) for some ε

$\varepsilon = 0.02$	0.05	0.10	0.15	0.20
$\gamma = 1.5260$	1.2573	1.0444	0.92184	0.83555

Tables of the Chebyshev polynomials [6] were used in calculating the function $p(x)$ by means of (3.3). Values of the quantity $aP^{-1}p(x)$ are presented in Table 2 for some ε and x/a.

The authors are grateful to L. P. Matveev and N. P. Chumykin for aid in performing the calculations.

REFERENCES

1. Galin, L. A., Contact Problems of Elasticity Theory. Gostekhteorizdat, Moscow, 1953.
2. Borodachev, N. M., Impression of a stamp with a narrow rectangular base into an elastic half-space. Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, №4, 1970.
3. Sherman, D. I., Integral equations method in two- and three-dimensional problems of static elasticity theory. In: Transactions All-Union Congress of Theoretical and Applied Mechanics. Izd. Akad. Nauk SSSR, Moscow-Leningrad, 1962.
4. Gradshtein, I. S. and Ryzhik, I. M., Tables of Integrals, Sums, Series and Products. "Nauka", Moscow, 1971.
5. Bateman, H. and Erdelyi, A., Higher Transcendental Functions, McGraw-Hill, New York, 1953.
6. Tables of the Chebyshev Polynomial $S_n(x)$ and $C_n(x)$. Computation Center Akad. Nauk SSSR, Moscow, 1963.

Translated by M. D. F.

UDC 539.3

AXISYMMETRIC STRAIN OF AN ELASTIC LAYER WITH A CIRCULAR LINE OF SEPARATION OF THE BOUNDARY CONDITIONS ON BOTH FACES

PMM Vol. 38, №1, 1974, pp. 131-138

V. N. ZAKORKO
(Komsomol'sk-on-Amur)

(Received April 28, 1973)

The problem of impressing a circular stamp into the upper face of a homogeneous elastic layer is considered. The layer rests on a stiff base weakened by a circular hole coaxial with the stamp and of the same radius. The surface of the stamp base possesses axial symmetry. The parts of the layer face outside the limits of contact are stress-free; there is no friction or cohesion between the layer and the stamp nor between the layer and the base.

A system of two integral equations with two unknown functions solving the problem is obtained by using the Hankel transform. This system is reduced successfully to Fredholm equations of the second kind. Approximate solutions of the equations are obtained for the flat stamp be the small parameter method.

Similar problems about a layer, but only with other boundary conditions, have been posed and solved in [1 – 4], say. But, insofar as the author knows, the boundary conditions in all such problems were assumed different only on one face of the layer and the problem was reduced by the integral transform method to one integral equation. Here the boundary conditions are "mixed" on both the upper and lower faces, hence, a system of two equations is obtained.

1. Let λ, μ be the elastic characteristics of a layer $0 \leqslant z \leqslant h$, $0 \leqslant x^2 + y^2 < \infty$; u_r, u_z are displacement components, and σ_r, σ_z, τ_{rz} are the stress components in a cylindrical coordinate system (see Fig. 1) in the axisymmetric case. The stamp radius is assumed to equal unity. Then the boundary conditions of the problem are:

$$\begin{aligned} u_z &= f(r), \quad \text{if} \quad z = h \text{ and } r \leqslant 1 \\ \sigma_z &= 0, \quad \text{if} \quad z = h \text{ and } r > 1 \\ u_z &= 0, \quad \text{if} \quad z = 0 \text{ and } r > 1 \\ \sigma_z &= 0, \quad \text{if} \quad z = 0 \text{ and } r < 1 \\ \tau_{rz} &= 0 \end{aligned}$$

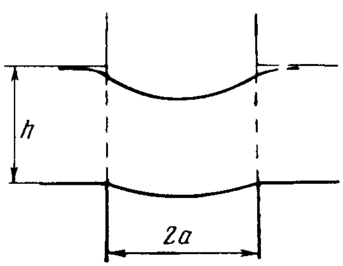

Fig. 1

Here $f(r)$ is a function defining the base of the stamp.

It is known [5] that in the presence of axial symmetry the strain and stress components are expressed in terms of one biharmonic function Φ

$$u_r = -\frac{\lambda+\mu}{\mu}\frac{\partial^2\Phi}{\partial r\,\partial z}, \qquad u_z = -\frac{\lambda+2\mu}{\mu}\nabla^2\Phi - \frac{\lambda+\mu}{\mu}\frac{\partial^2\Phi}{\partial z^2} \tag{1.1}$$

$$\sigma_r = \lambda\nabla^2\Phi - 2(\lambda+\mu)\frac{\partial^3\Phi}{\partial z^3}$$

$$\sigma_z = (3\lambda+4\mu)\nabla^2\frac{\partial^2\Phi}{\partial z^2} - 2(\lambda+\mu)\frac{\partial^3\Phi}{\partial z^3}$$

$$\tau_{rz} = (\lambda+2\mu)\frac{\partial}{\partial z}\nabla^2\Phi - 2(\lambda+\mu)\frac{\partial^3\Phi}{\partial z^2\,\partial r}$$

Let us take the general solution of the biharmonic equation in the form of a Hankel transform

$$\Phi(r, z) = \int_0^\infty P(\gamma, z) J_0(\gamma z)\gamma\, d\gamma \tag{1.2}$$

$$P(\gamma, z) = (A + Bz)\, e^{-\gamma z} + (C + Dz)\, e^{\gamma z}$$

The boundary condition $\tau_{rz} = 0$ at $z = 0$ and $z = h$ permits elimination of two out of the four functions

$$A = A(\gamma), \ldots, D = D(\gamma)$$

After elementary computations, we obtain by using the last equalities in (1.1) and (1.2)

$$\begin{aligned} P(\gamma, z) &= A\,[(1 + az)\,\text{ch}\,\gamma z - (1 - \gamma kz)\,\text{sh}\,\gamma z] + \\ &\quad C\,[(1 + bz)\,ch\,\gamma z + (1 - \gamma kz)\,\text{sh}\,\gamma z] \end{aligned} \tag{1.3}$$

$$k = 1 + \mu/\lambda, \quad a = \frac{k\gamma}{\Delta}(1 + \gamma kz)\,\mathrm{sh}\,\gamma h$$

$$b = \frac{k\gamma}{\Delta}(\gamma kz - 1)\,\mathrm{sh}\,\gamma h, \qquad \Delta = \gamma kz\,\mathrm{ch}\,\gamma h + \mathrm{sh}\,\gamma h$$

Now, we substitute its representation (1.2), (1.3) instead of $\Phi(r, z)$ in the right sides of the second and fourth equations in (1.1). We have

$$u_z(r, z) = \int_0^\infty \{A[-k(3 + az)\,\mathrm{ch}\,\gamma z + (2a + k\gamma + k^2\gamma^2 z)\,\mathrm{sh}\,\gamma z] + \tag{1.4}$$

$$C[-k(3 + bz)\,\mathrm{ch}\,\gamma z + (2b - k\gamma + k^2\gamma^2 z)\,\mathrm{sh}\,\gamma z]\}\,J_0(\gamma r)\,\gamma^2\,d\gamma$$

$$\sigma_z(r, z) = 2\lambda \int_0^\infty \{A[(k\gamma + (k - 1)\,a + k^2\gamma^2 z)\,\mathrm{ch}\,\gamma z - k\gamma\,(k - az)\,\mathrm{sh}\,\gamma z] +$$

$$C[(-k\gamma + (k - 1)\,b + k^2\gamma^2 z)\,\mathrm{ch}\,\gamma z - k\gamma\,(k + bz)\,\mathrm{sh}\,\gamma z]\}\,J_0(\gamma r)\,\gamma^3\,d\gamma$$

Keeping in mind the equality [6]

$$\int_0^\infty \sin(\gamma t)\,J_0(\gamma r)\,d\gamma = \begin{cases} 0, & r > t \\ \dfrac{1}{\sqrt{t^2 - r^2}}, & r < t \end{cases} \tag{1.5}$$

we represent $u_z(r, 0)$ as an integral containing a new unknown function $\varphi(t)$

$$u_z(r, 0) = \int_0^1 \varphi(t)\,dt \int_0^\infty \sin(\gamma t)\,J_0(\gamma r)\,d\gamma \tag{1.6}$$

In this case, the boundary condition $u_z(r, 0) = 0$ for $r > 1$ is satisfied automatically by virtue of (1.5). If $r < 1$, then

$$u_z(r, 0) = \int_r^1 \frac{\varphi(t)}{\sqrt{t^2 - r^2}}\,dt$$

For the settling under the stamp to be bounded, $\varphi(0) = 0$ must be assumed and this condition must henceforth be satisfied. If it were also assumed that $\varphi'(t)$ is continuous on the segment [0, 1], then

$$u_z(r, 0) = \varphi(1)\ln(1 + \sqrt{1 - r^2}) - \int_r^1 \varphi'(t)\ln(t + \sqrt{t^2 - r^2})\,dt \tag{1.7}$$

Changing the order of integration in the integral in (1.6) and setting $z = 0$ in the first equation in (1.4), we arrive at the equation

$$-3k\gamma^3(A + C) = \int_0^1 \varphi(t)\sin(\gamma t)\,dt \tag{1.8}$$

In exactly the same way, in order to satisfy the boundary condition $\sigma_z(r, h) = 0$ for $r > 1$, we set

$$\frac{1}{2\lambda}\sigma_z(r, h) = \int_0^1 \psi(t)\,dt \int_0^\infty \cos(\gamma t)\,\gamma J_0(\gamma r)\,d\gamma \tag{1.9}$$

where $\psi(t)$ is a new unknown function, wherein we assume that it is also continuously differentiable. We have

$$\frac{1}{2\lambda}\sigma_z(r, h) = \int_0^\infty J_0(\gamma r)\,\gamma\,d\gamma \int_0^1 \psi(t)\cos(\gamma t)\,dt =$$

$$\psi(1)\int_0^\infty J_0(\gamma r)\sin t\,dt-\int_0^1\psi'(t)\,dt\int_0^\infty J_0(\gamma r)\sin(\gamma t)\,d\gamma$$

By virtue of (1.5) the boundary condition $\sigma_z(r, h) = 0$ for $r > 1$ is satisfied automatically. If $r < 1$, then

$$\frac{1}{2\lambda}\sigma_z(r, h)=\frac{\psi(1)}{\sqrt{1-r^2}}-\int_r^1\frac{\psi'(t)}{\sqrt{t^2-r^2}}\,dt \tag{1.10}$$

Let us set $z = h$ into the second formula in (1.4), and let us change the order of integration in the right side of (1.8). We then obtain

$$\begin{gathered}\gamma^3\{[(k-1)a+k\gamma+\gamma^2k^2h]\operatorname{ch}(\gamma h)-\gamma h(k+ah)\operatorname{sh}(\gamma h)\}A+ \\ \gamma^3\{[(k-1)b-k\gamma+\gamma^2k^2h]\operatorname{ch}(\gamma h)-\gamma h(k+bh)\operatorname{sh}(\gamma h)\}C= \\ \int_0^1\psi(t)\sin(\gamma t)\,dt\end{gathered} \tag{1.11}$$

Now, we solve the system (1, 8), (1, 11) for A and C. We substitute the values found into the right sides of (1.4), and setting $z = h$ and $z = 0$ therein, respectively, we require compliance with the boundary conditions $\sigma_z(0, r) = 0$, $u_z(h, r) = f(r)$ for $r < 1$.

We then obtain a system of integral equations in the unknowns $\varphi(t)$ and $\psi(t)$

$$\int_0^\infty[\delta_{11}(\gamma)\Phi(\gamma)+\delta_{12}(\gamma)\Psi(\gamma)]J_0(\gamma r)\,d\gamma=0 \tag{1.12}$$

$$\int_0^\infty[\delta_{21}(\gamma)\Phi(\gamma)+\delta_{22}(\gamma)\Psi(\gamma)]J_0(\gamma r)\,d\gamma=f(r)$$

Here

$$\Phi(\gamma)=\int_0^1\varphi(t)\sin(\gamma t)\,dt=-\varphi(1)\frac{\cos\gamma}{\gamma}+\frac{1}{\gamma}\int_0^1\varphi'(t)\cos(\gamma t)\,dt \tag{1.13}$$

$$\Psi(\gamma)=\int_0^1\psi(t)\cos(\gamma t)\,dt=\psi(1)\frac{\sin\gamma}{\gamma}-\frac{1}{\gamma}\int_0^1\psi'(t)\sin(\gamma t)\,dt$$

$$\delta_{11}(\gamma)=-\frac{k\gamma}{3\Delta_0}(\operatorname{sh}^2\gamma h-\gamma^2h^2),\qquad \delta_{12}(\gamma)=\frac{\gamma}{\Delta_0}(\gamma h\operatorname{ch}\gamma h+\operatorname{sh}\gamma h)$$

$$\delta_{21}(\gamma)=\frac{1}{\Delta_0}(\gamma h\operatorname{ch}\gamma h+\operatorname{sh}\gamma h),\qquad \delta_{22}(\gamma)=-\frac{1}{k\Delta_0}\operatorname{sh}^2\gamma h$$

$$\Delta_0=\operatorname{sh}(\gamma h)\operatorname{ch}(\gamma h)+\gamma h,\qquad k=1+\mu/\lambda$$

The following asymptotic formulas can be obtained for $\delta_{ij}(\gamma)$ $(i, j = 1, 2)$ as $\gamma h\to+\infty$:

$$\delta_{11}(\gamma)\sim-\frac{k}{3}\gamma-\frac{4k}{3}\gamma^3h^2e^{-2\gamma h},\qquad \delta_{12}(\gamma)\sim2\gamma^2e^{-\gamma h} \tag{1.14}$$

$$\delta_{21}(\gamma)\sim2\gamma he^{-\gamma h},\qquad \delta_{22}(\gamma)\sim\frac{-1}{k}+\frac{4\gamma h}{k}e^{-2\gamma h}$$

Taking account of these formulas, it can be shown that the integrals in the left sides of (1.12) converge for any $r > 0$.

The problem evidently reduces to solving the system (1.12). In particular, if $\varphi(t)$ and $\psi(t)$ are found. We determine the stress at the stamp contact by (1.10) and the layer displacement over the hole by (1.7). The stresses and displacements within the layer are determined by (1.4) in which the functions $A(\gamma)$ and $C(\gamma)$ are easily expressed in terms of φ and ψ. Obtaining numerical results can certain turn out to be a tedious job.

2. Let us return to the system (1.2). If their integral representations (1.13) are substituted instead of $\Phi(\gamma)$ and $\Psi(\gamma)$ in (1.12), and the order of integration is changed (the validity of this operation can be given a foundation), then this system can be represented as follows:

$$\frac{k}{3}\int_0^r \frac{\varphi'(t)}{\sqrt{r^2-t^2}}\,dt = \varphi(1)K_{11}(r,1)+\psi(1)K_{12}(r,1)- \tag{2.1}$$

$$\int_0^1 [\varphi'(t)K_{11}(r,t)+\psi'(t)K_{12}(r,t)]\,dt$$

$$\frac{1}{k}\int_0^r \frac{t\psi'(t)}{r\sqrt{r^2-t^2}}\,dt = \varphi(1)K_{21}(r,1)+\psi(1)K_{22}(r,1)-$$

$$\int_0^1 [\varphi'(t)K_{21}(r,t)+\psi'(t)K_{22}(r,t)]\,dt - f'(r)$$

Here

$$K_{11}(r,t)=\int_0^\infty P_{11}(\gamma)\cos(\gamma t)J_0(\gamma r)\,d\gamma, \qquad P_{11}(\gamma)=-\frac{\delta_{11}}{\gamma}-\frac{k}{3}$$

$$K_{12}(r,t)=\int_0^\infty P_{12}(\gamma)\sin(\gamma t)J_0(\gamma r)\,d\gamma, \qquad P_{12}(\gamma)=\frac{\delta_{12}}{\gamma}$$

$$K_{21}(r,t)=\int_0^\infty P_{21}(\gamma)\cos(\gamma t)J_1(\gamma r)\,d\gamma, \qquad P_{21}(\gamma)=\delta_{21}$$

$$K_{22}(r,t)=\int_0^\infty P_{22}(\gamma)\sin(\gamma t)J_1(\gamma r)\,d\gamma, \qquad P_{22}(\gamma)=-\delta_{22}-\frac{1}{k}$$

The second equation in the system (1.12) was differentiated with respect to r during the manipulation, hence, the function $f'(r)$ appeared in the right side of the equality in the system (2.1).

The singularities inherent in the kernels of the equations in the system (1.12) have here already been extracted explicitly, and they are in the left sides of (2.1). The kernels in the right sides $K_{ij}(r,t)$ $(i, j = 1, 2)$ are continuous functions in the square $[0, 1]\times[0, 1]$. Indeed, it follows from (1.14) that $P_{ij}(\gamma)$ tends to zero as $\gamma h \to +\infty$, at least as $\gamma h e^{-\gamma h}$, and a conclusion about the continuity of the kernels can hence be deduced.

The inverses are known for the operators in the left sides of (2.1); if the right sides of (2.1) are temporarily considered known, and we set $t = r\sin\theta$, we obtain the Schloemilch equation

$$\int_0^{\pi/2} F(r\sin\theta)\,d\theta = g(r)$$

whose continuous solutions are [1]

$$F(x) = \frac{2}{\pi}\left[g(0) + x\int_0^{\pi/2} g'(x\sin\theta)\,d\theta\right]$$

Since the right sides of (2.1) are actually unknown, this formula generates new integral equations. Omitting the elementary but awkward manipulations, let us present the results (for the case of a flat stamp $f'(r) = 0$)

$$\varphi'(x) = A\Pi_{11}(1, x) + B\Pi_{12}(1, x) - \int_0^1 [\varphi'(t)\,\Pi_{11}(t, x) + \psi'(t)\,\Pi_{12}(t, x)]\,dt \tag{2.2}$$

$$\psi'(t) = A\Pi_{21}(1, x) + B\Pi_{22}(1, x) - \int_0^1 [\varphi'(t)\,\Pi_{21}(t, x) + \psi'(t)\,\Pi_{22}(t, x)]\,dt$$

Here

$$A = \varphi(1), \quad B = \psi(1) \tag{2.3}$$

$$\Pi_{11}(t, x) = \frac{2}{\pi}\int_0^\infty \frac{3}{k} P_{11}(\gamma)\cos(\gamma t)\cos(\gamma x)\,d\gamma \tag{2.4}$$

$$\Pi_{12}(t, x) = \frac{2}{\pi}\int_0^\infty \frac{3}{k} P_{12}(\gamma)\sin(\gamma t)\cos(\gamma x)\,d\gamma$$

$$\Pi_{21}(t, x) = \frac{2}{\pi}\int_0^\infty kP_{21}(\gamma)\cos(\gamma t)\sin(\gamma x)\,d\gamma$$

$$\Pi_{22}(t, x) = \frac{2}{\pi}\int_0^\infty kP_{22}(\gamma)\sin(\gamma t)\sin(\gamma x)\,d\gamma$$

Just as the kernels K_{ij}, the kernels $\Pi_{ij}(t, x)$ are continuous in the square $[0, 1]\times[0, 1]$, hence (2.2) are Fredholm equations.

The two arbitrary constants A and B in the system (2.2) are determined as follows. We find the solution of the system (2.2) as

$$\varphi'(t) = A\varphi_1(t) + B\varphi_2(t), \qquad \psi'(t) = A\psi_1(t) + B\psi_2(t)$$

In order to satisfy the condition $\varphi(0) = 0$ imposed earlier, we must assume

$$\varphi(x) = A\int_0^x \varphi_1(t)\,dt + B\int_0^x \varphi_2(t)\,dt$$

$$\psi(x) = A\int_0^x \psi_1(t)\,dt + B\int_0^x \psi_2(t)\,dt + C$$

where C is the constant of integration.

The conditions (2.3) reduce to a system of two equations in A, B and C. We determine A and B from this system to the accuracy of the factor C : $A = CA_0$, $B = CB_0$. Therefore

$$\psi(x) = C\left[A_0\int_0^x \psi_1(x)\,dx + B_0\int_0^x \psi_2(x)\,dx\right] = C\psi_0(x)$$

Finally, in order to determine C, let us evaluate the pressure on the stamp and let us equate it to the external force P. According to (1.10) we have

$$P = \iint\limits_{(0\leqslant x^2+y^2\leqslant 1)} \sigma_z\,dx\,dy = 4\pi\lambda\int_0^1 r\left[\frac{\psi(1)}{\sqrt{1-r^2}} - \int_r^1 \frac{\psi'(t)}{\sqrt{t^2-r^2}}\,dt\right]dr =$$

$$4\pi\lambda\left[\psi(1) - \int_0^1 \psi'(t)\,dt\int_0^t \frac{r\,dr}{\sqrt{t^2-r^2}}\right] = 4\pi\lambda\int_0^1 \psi(t)\,dt =$$

$$4\pi\lambda C\int_0^1 \psi_0(t)\,dt$$

3. The equations (2.2) can be solved by different approximate methods. The simplest is probably the small parameter method. If we set $\gamma h = u$ and then $h^{-1} = p$ (therefore p is the ratio of the stamp radius to the layer thickness), then the kernels $\Pi_{ij}(x, t)$ are easily expanded in a power series in p.

For example

$$\Pi_{11}(x, t) = \frac{2}{\pi}\int_0^\infty \frac{3}{k} P_{11}(\gamma)\,p\cos(ptu)\cos(pxu)\,du = \qquad (3.1)$$

$$\frac{3}{\pi k}\int_0^\infty P_{11}(\gamma)\,p\,[\cos p(x+t)u + \cos p(x-t)u]\,du =$$

$$pa_1 + p^3a_3 + p^5a_5 + \ldots + p^{2n+1}a_{2n+1} + \ldots$$

$$a_{2n+1} = \frac{(-1)^n}{(2n)!}\frac{6}{\pi k} M_{2n}(x, t)\int_0^\infty P_{11}(u)\,u^{2n}\,du$$

$$M_{2n}(x, t) = \frac{1}{2}\left[(t+x)^{2n} + (t-x)^{2n}\right]$$

If $|p| < 1$, the convergence of the series (3.1) can be proved by using the relationships

$$|P_{11}(u)| < Mu^2e^{-2u}\ (M = \text{const}), \qquad |M_{2n}(x, t)| < 2^{2n-1} + 1$$

$$\int_0^\infty u^{2n+2}e^{-2u}\,du = \frac{(2n+2)!}{2^{2n+1}}$$

An analogous expansion is obtained in the domain $|p| < 1$ for the kernel $\Pi_{22}(x, t)$.

Similar expansions are valid for the kernels $\Pi_{12}(x, t)$ and $\Pi_{21}(x, t)$ only in the domain $|p| < {}^1/_2$. This is associated with the fact that the functions $P_{12}(\gamma)$ and $P_{21}(\gamma)$ have different asymptotics than the functions $P_{11}(\gamma)$ and $P_{22}(\gamma)$ for infinitely large $u = \gamma h$.

Evaluation of the integrals

$$\int_0^\infty P_{ij} u^m \, du \equiv f_{ij}(m)$$

was carried out on the "Iskra-11" electronic computer for the first three values of the exponent m. The range of integration was divided into two parts: $[0, 6]$ and $[6, \infty)$. The integral over the finite part was computed by the Simpson formula with 48 nodes while the functions P_{ij} were approximated in the infinite part by simpler functions, mostly elementary ones. The approximation was accomplished every time so that the integrals were calculated to 0.001 accuracy.

The results of the calculations are $(f_{21}(m) = f_{12}(m))$

	$m = 0$	$m = 2$	$m = 4$
$f_{11}(m)$	1.944	4.247	29.219
$f_{22}(m)$	1.168	1.582	8.238
	$m = 0$	$m = 1$	$m = 3$
$f_{12}(m)$	2.892	5.169	58.344

In conformity with this, we have (for $\lambda = \mu$)

$$\begin{aligned}
\Pi_{11}(t, x) &= -1.237\, p + 1.352\,(x^2 + t^2)p^3 - 0.775\,(x^4 + 6x^2t^2 + t^4)\,p^5 + \ldots \\
\Pi_{12}(t, x) &= 4.929\, p^2 - 9.273\,(t^3 - 3tx^2)p^4 + \ldots \\
\Pi_{21}(t, x) &= 6.572p^2 - 12.364\,(x^3 + 3x^2t)\,p^4 + \ldots \\
\Pi_{22}(t, x) &= -0.5032\, xtp^3 + 0.218\,(4t^2x + 4tx^2)\,p^5 - \ldots
\end{aligned}$$

It is evidently sufficient that

$$\begin{aligned}
&\alpha = \max\left(\|\Pi_{11}\| + \|\Pi_{21}\|,\ \|\Pi_{12}\| + \|\Pi_{22}\|\right) < 1 \\
&(\|\Pi_{ij}\| = \max |\Pi_{ij}(x, t)|) \\
&(t, x) \in [0.1] \times [0.1]
\end{aligned}$$

for the iteration process to converge.

Using the values of the integrals $f_{ij}(m)$ it can be shown that $\alpha < 1$ if $|p| < 0.3$. Under this condition the successive approximations converge uniformly in the domain $|p| < 0.3$, and since these approximations are analytic functions of p in the domain mentioned, then the solution is also analytic in p because of the known Weierstrass theorem. Hence, in the case $|p| < 0.3$ the solution of the problem can be sought at once as a power series in p.

Therefore, some of the first terms of the expansions of the desired functions have been obtained

$$\begin{aligned}
\varphi'(x) = {} & A\,[-1.238\, p - 1.532\, p^2 + (1.352\, x^2 - 0.545)\, p^3 + (1.674\, x^2 - 10.343)\, p^4 - \ldots] + \\
& B\,[4.929\, p^2 + 6.101\, p^3 - (27.819\, x^2 + 1.722)\, p^4 - \ldots] \\
\psi'(x) = {} & A\,[4.929\, p^2 + 6.101\, p^3 - (27.819 x^2 + 1.722)\, p^4 - \ldots] + \\
& B\,[1.058\, xp^2 - 32.392\, xp^3 + (0.873\, x^3 - 19.222\, x)\, p^4 + \ldots]
\end{aligned}$$

REFERENCES

1. Ufliand, Ia. S., Integral Transforms in Elasticity Theory Problems. "Nauka", Leningrad, 1967.
2. Aleksandrov, V. M., On the approximate solution of some integral equations of the theory of elasticity and mathematical physics. PMM Vol. 31, №6, 1967.
3. Aleksandrov, V. M., Asymptotic methods in contact problems of elasticity theory. PMM Vol. 32, №4, 1968.
4. Aleksandrov, V. M., Babeshko, V. A. and Kucherov, V. A., Contact problems for an elastic layer of slight thickness. PMM Vol. 30, №1, 1966.
5. Sneddon, I., Fourier Transforms. Moscow, Izd. Inostr. Lit., 1955.
6. Gradshtein, I. S. and Ryzhik, I. I., Tables of Integrals, Sums, Series and Products. Fizmatgiz, Moscow, 1967.

Translated by M. D. F.

UDC 539.3

ON THE LOCAL AXISYMMETRIC COMPRESSION OF AN ELASTIC LAYER WEAKENED BY AN ANNULAR OR CIRCULAR CRACK

PMM Vol. 38, №1, 1974, pp. 139-144
V. S. NIKISHIN and G. S. SHAPIRO
(Moscow)
(Received May 8, 1973)

Two kindred problems on the compression of an elastic layer by a local load applied symmetrically to its surfaces are considered.

In one case the layer has an annular crack with inner radius a and outer radius b on the middle plane. The quantities a and b $(0 < a < b)$ are selected from the condition that the annular crack subjected to a load would be opened up and a normal tensile stress concentration would originate on the circumferential contours $r = a$ and $r = b$.

In the other case, the layer has a circular crack of radius b on the middle plane. Under the effect of a load in a circular domain of radius a $(a < b)$ the crack edges will be in contact, and will separate from each other in the annular region $a < r < b$. The quantity a is unknown and to be determined from the condition that the contact pressure on the circumferential contour $r = a$ is zero; the quantity b is selected from the condition that a normal tensile stress concentration would originate on the contour $r = b$.

In both cases the crack lips are assumed smooth. The crack is a mathematical slit in the unloaded layer.

In the general case, the layer is compressed under the effect of an arbitrary local load applied to its upper and lower boundary planes symmetrically relative to the axis and the middle plane. As an illustration, the particular case of compression of the layer by two normal concentrated forces directed along the axis of symmetry of the problem is considered (Fig. 1).

The problems of annular and circular cracks in an infinite layer were considered